Triple Integration

Example

Integrate the following:

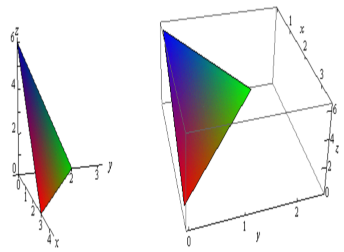
$\iiint_B (8xyz)dV$ where $B = [2,3] \times [1,2] \times [0,1]$

$\int_2^3 \int_1^2 \int_0^1 (8xyz)dxdydz = 15$

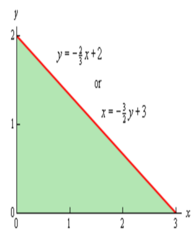
Example

Integrate the following:

$\iiint_E (2x)dV$ where E is the region under the plane $2x + 2y + z = 6$ in the first octant



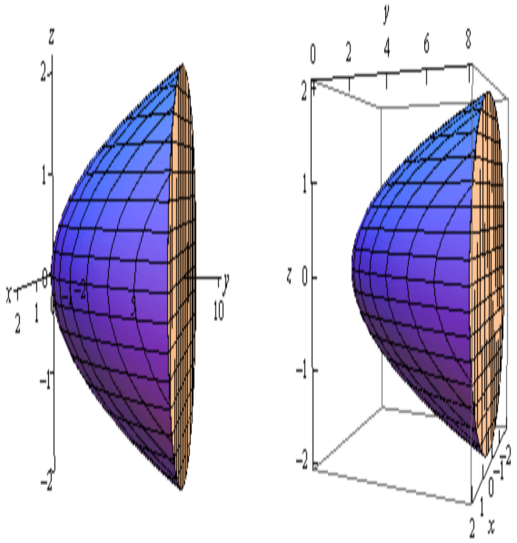
We now need to determine the region D in the xy-plane. We can get a visualization of the region by pretending to look straight down on the object from above. What we see will be the region D in the xy-plane. So D will be the triangle with vertices at $(0,0)$, $(3,0)$, and $(0,2)$. Here is a sketch of D.



$\int_0^3 \int_0^{\frac{-2}{3}x+2} \int_0^{6-2x-2y} 2xdzdydx = 9$

Example

Evaluation $\iiint_E \sqrt{3x^2 + 3z^2}dV$ where E is the solid bounded by the plane $y = 2x^2 + 2z^2$ and y=8

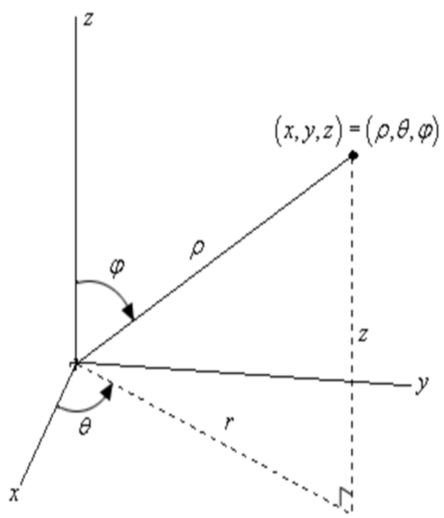


$$\iint \int_{3x^2+3z^2}^{8} \sqrt{3x^2+3z^2}\,dy\,dx\,dz$$

$$\iint \sqrt{3x^2+3z^2}\left(8-(2x^2+2z^2)\right)dA$$

$$\sqrt{3}\int_0^{2\pi}\int_0^2 (8r-2r^3)\,r\,dr\,d\theta = \frac{256\sqrt{3}\pi}{15}$$

Spherical Coordinates



Finding the jacobian of spherical coordinates.

$$x = \rho \sin\varphi \cos\theta \quad y = \rho \sin\varphi \sin\theta \quad z = \rho \cos\varphi$$

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} &= \begin{vmatrix} \sin\varphi\cos\theta & -\rho\sin\varphi\sin\theta & \rho\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & \rho\sin\varphi\cos\theta & \rho\cos\varphi\sin\theta \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix} \\
&= -\rho^2\sin^3\varphi\cos^2\theta - \rho^2\sin\varphi\cos^2\varphi\sin^2\theta + 0 \\
&\quad - \rho^2\sin^3\varphi\sin^2\theta - 0 - \rho^2\sin\varphi\cos^2\varphi\cos^2\theta \\
&= -\rho^2\sin^3\varphi\left(\cos^2\theta + \sin^2\theta\right) - \rho^2\sin\varphi\cos^2\varphi\left(\sin^2\theta + \cos^2\theta\right) \\
&= -\rho^2\sin^3\varphi - \rho^2\sin\varphi\cos^2\varphi \\
&= -\rho^2\sin\varphi\left(\sin^2\varphi + \cos^2\varphi\right) \\
&= -\rho^2\sin\varphi
\end{aligned}$$

$$dV = \left|-\rho^2\sin\varphi\right| \, d\rho \, d\theta \, d\varphi = \rho^2\sin\varphi \, d\rho \, d\theta \, d\varphi$$

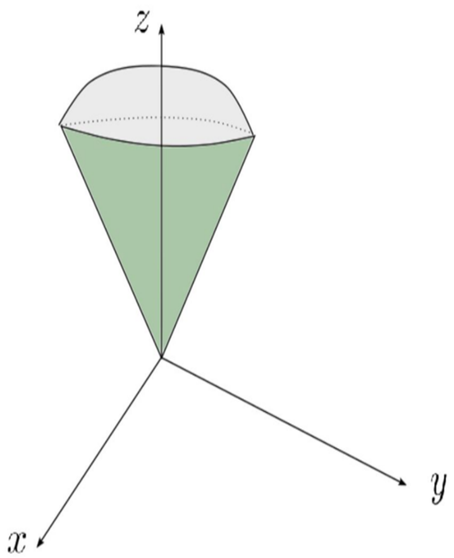
Example

Evaluate $\iiint_E 16zdV$ where E is the upper half of the sphere $x^2 + y^2 + z^2 = 1$

$\iiint 16zdV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 sin(\phi)(16\rho cos(\phi))d\rho d\theta d\phi = 4\pi$

Example

Evaluate the Volume D that lies inside the cone $\phi = \frac{1}{6}$ and in the sphere $\rho = 4$.



$V = \int_0^4 \int_0^{\frac{\pi}{6}} \int_0^{2\pi} \rho^2 sin(\phi)d\rho d\theta d\phi = \frac{128\pi}{3} - \frac{64\pi\sqrt{3}}{3}$

Example

Evaluate $\iiint zxdV$ where E is inside $x^2 + y^2 + z^2 = 4$ and the cone (pointing upward) that makes an angle of $\frac{\pi}{3}$ with the negative z-axis and has $x \leq 0$

$$\iiint 16zdV = \int_0^2 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\frac{2\pi}{3}}^{\pi} (\rho cos(\phi))(\rho sin(\phi)cos(\theta))\rho^2 sin(\phi)d\rho d\theta d\phi = \frac{8\sqrt{3}}{5}$$