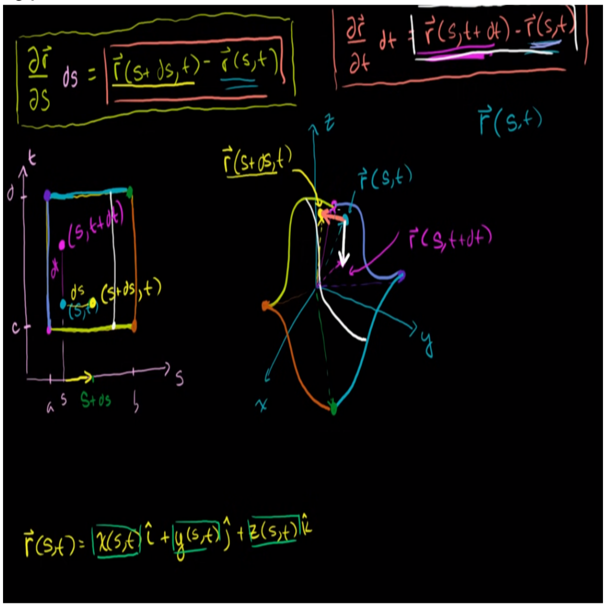
Surface Integrals

Background

$\frac{\partial \vec{r}}{\partial s} ds = \vec{r}(s+ds,t) - \vec{r}(s,t)$

$\frac{\partial \vec{r}}{\partial t} dt = \vec{r}(s,t+dt) - \vec{r}(s,t)$

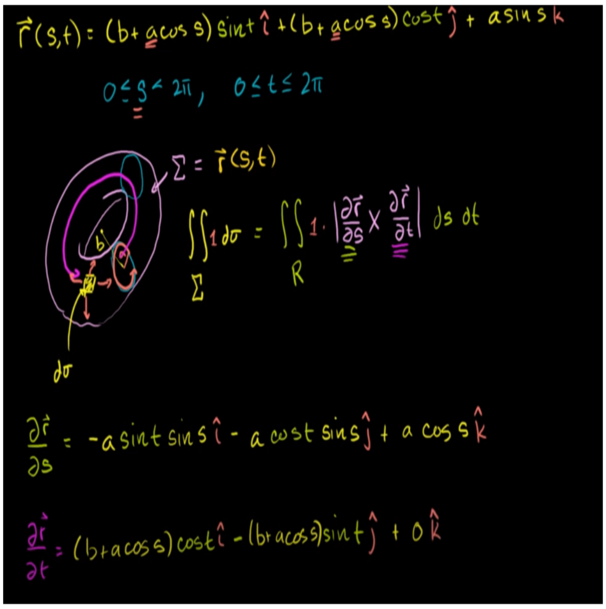


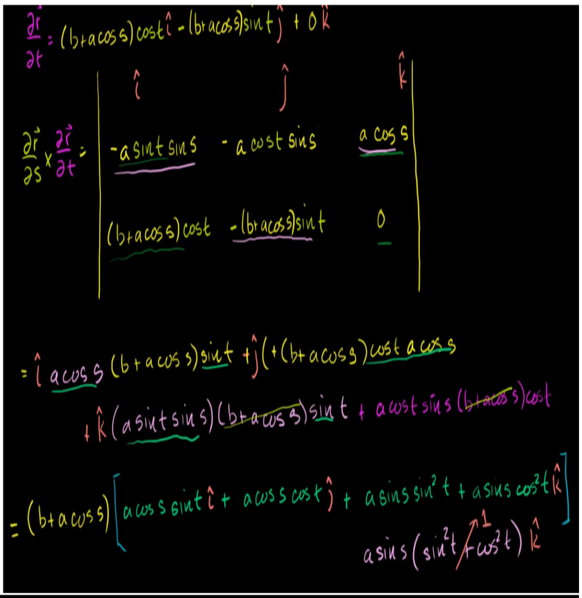
$\iint_{\Sigma} d\sigma$ \)

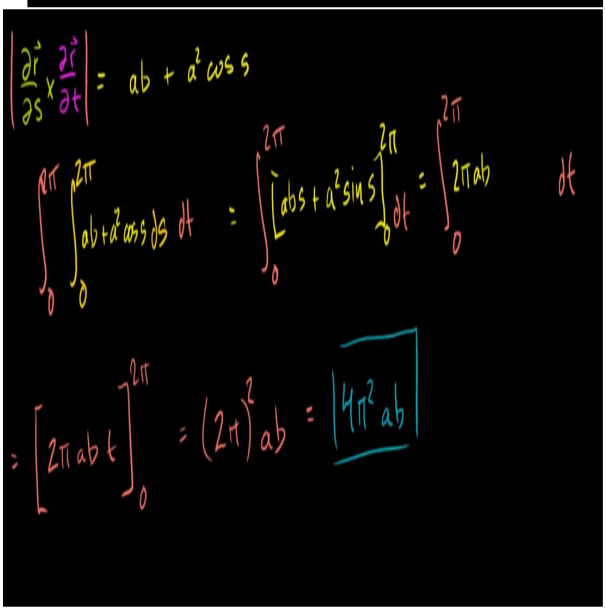
$\iint_{A} |\frac{\partial \vec{r}}{\partial s} ds \times \frac{\partial \vec{r}}{\partial t} dt|$

$\iint_{A} |\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}| ds dt$

Example - torus







Example - Sphere

$$\vec{r}(\phi,\theta) = a\sin(\phi)\cos(\theta)\hat{\imath} + a\sin(\phi)\sin(\theta)\hat{\jmath} + a\cos(\phi)\hat{k}$$

$$0 \le \phi \le \pi$$

$$0 \le \theta \le 2\pi$$

$$\vec{r}_\phi = a\cos(\phi)\cos(\theta)\hat{\imath} + a\cos(\phi)\sin(\theta)\hat{\jmath} - a\sin(\phi)\hat{k}$$

$$\vec{r}_\theta = -a\sin(\phi)\sin(\theta)\hat{\imath} + a\sin(\phi)\cos(\theta)\hat{\jmath} + 0\hat{k}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a\cos(\phi)\cos(\theta) & a\cos(\phi)\sin(\theta) & -a\sin(\phi) \\ -a\sin(\phi)\sin(\theta) & a\sin(\phi)\cos(\theta) & 0 \end{vmatrix}$$

$$A = \int_0^{2\pi}\int_0^{\pi} a^2\sin(\phi)\,d\phi\,d\theta$$

$$= a^2\int_0^{2\pi} -\cos(\phi)\Big|_0^{\pi} d\theta$$

$$= a^2\int_0^{2\pi} 2\,d\theta = 4\pi a^2$$

Example

$$z = g(x, y) \qquad \vec{r} = \langle x, y, z(x, y) \rangle$$

$$\vec{r}_x = \langle 1, 0, \partial z/\partial x \rangle \qquad \vec{r}_y = \langle 0, 1, \partial z/\partial y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \partial z/\partial x \\ 0 & 1 & \partial z/\partial y \end{vmatrix} = -\frac{\partial z}{\partial x}\hat{i} - \frac{\partial z}{\partial y}\hat{j} + \hat{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1}$$

$$\iint f[x, y, z(x, y)] \sqrt{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1}\, dx\, dy$$

$$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(\sin v)^2 + (-\cos v)^2 + u^2} = \sqrt{1 + u^2}$$

$$dS = |\vec{r}_u \times \vec{r}_v|\, du\, dv = \sqrt{1 + u^2}\, du\, dv$$

$$\iint_S f\, dS = \iint f \sqrt{1 + u^2}\, du\, dv$$

Example

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