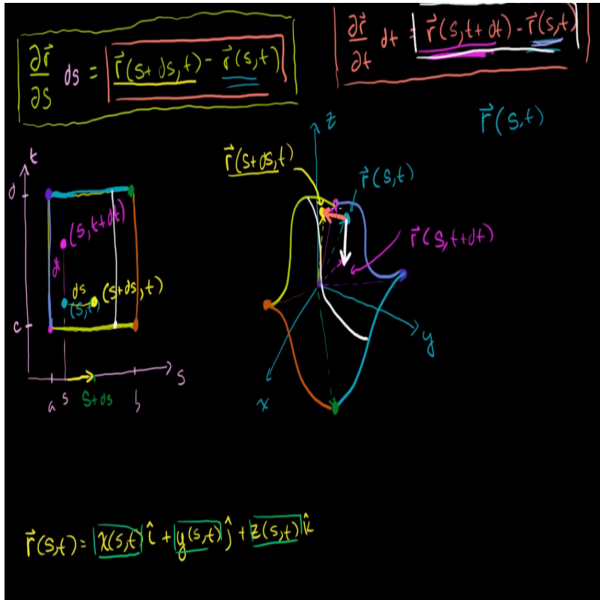


# Surface Integrals

## Background

$$\frac{\partial \vec{r}}{\partial s} ds = \vec{r}(s + ds, t) - \vec{r}(s, t)$$

$$\frac{\partial \vec{r}}{\partial t} dt = \vec{r}(s, t + dt) - \vec{r}(s, t)$$



$$\iint_{\Sigma} d\sigma \setminus$$

$$\iint_A \left| \frac{\partial \vec{r}}{\partial s} ds \times \frac{\partial \vec{r}}{\partial t} dt \right|$$

$$\iint_A \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right| ds dt$$

## Example - torus

$$\vec{r}(s,t) = (b + a \cos s) \sin t \hat{i} + (b + a \cos s) \cos t \hat{j} + a \sin s \hat{k}$$

$$0 \leq s < 2\pi, \quad 0 \leq t \leq 2\pi$$

$$\Sigma = \vec{r}(s,t)$$

$$\iint_{\Sigma} 1 \, d\sigma = \iint_R 1 \cdot \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right| ds \, dt$$

$$\frac{\partial \vec{r}}{\partial s} = -a \sin t \sin s \hat{i} - a \cos t \sin s \hat{j} + a \cos s \hat{k}$$

$$\frac{\partial \vec{r}}{\partial t} = (b + a \cos s) \cos t \hat{i} - (b + a \cos s) \sin t \hat{j} + 0 \hat{k}$$

$$\frac{\partial \vec{r}}{\partial t} = (b + a \cos s) \cos t \hat{i} - (b + a \cos s) \sin t \hat{j} + 0 \hat{k}$$

$$\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a \sin t \sin s & -a \cos t \sin s & a \cos s \\ (b + a \cos s) \cos t & -(b + a \cos s) \sin t & 0 \end{vmatrix}$$

$$= \hat{i} a \cos s (b + a \cos s) \sin t \hat{j} + (b + a \cos s) \cos t a \cos s \hat{k} + \hat{k} (a \sin t \sin s) (b + a \cos s) \sin t + a \cos t \sin s (b + a \cos s) \cos t$$

$$= (b + a \cos s) \left[ a \cos s \sin t \hat{i} + a \cos s \cos t \hat{j} + a \sin s \sin^2 t + a \sin s \cos^2 t \hat{k} \right]$$

$$\left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right| = ab + a^2 \cos s$$

$$\int_0^{2\pi} \int_0^{2\pi} (ab + a^2 \cos s) ds \, dt = \int_0^{2\pi} \left[ abs + a^2 \sin s \right]_0^{2\pi} dt = \int_0^{2\pi} 2\pi ab \, dt$$

$$= \left[ 2\pi ab t \right]_0^{2\pi} = (2\pi)^2 ab = \boxed{4\pi^2 ab}$$

**Example - Sphere**

$$\vec{r}(\phi, \theta) = a \sin(\phi) \cos(\theta) \hat{i} + a \sin(\phi) \sin(\theta) \hat{j} + a \cos(\phi) \hat{k}$$

$$0 \leq \phi \leq \pi$$

$$0 \leq \theta \leq 2\pi$$

$$\vec{r}_\phi = a \cos(\phi) \cos(\theta) \hat{i} + a \cos(\phi) \sin(\theta) \hat{j} - a \sin(\phi) \hat{k}$$

$$\vec{r}_\theta = -a \sin(\phi) \sin(\theta) \hat{i} + a \sin(\phi) \cos(\theta) \hat{j} + 0 \hat{k}$$

$$\vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos(\phi) \cos(\theta) & a \cos(\phi) \sin(\theta) & -a \sin(\phi) \\ -a \sin(\phi) \sin(\theta) & a \sin(\phi) \cos(\theta) & 0 \end{vmatrix}$$

SUBSCRIBE

$$A = \int_0^{2\pi} \int_0^\pi a^2 \sin(\phi) d\phi d\theta$$

$$= a^2 \int_0^{2\pi} -\cos(\phi) \Big|_0^\pi d\theta$$

$$= a^2 \int_0^{2\pi} 2 d\theta = 4\pi a^2$$

## Example

$$z = g(x, y) \quad \vec{r} = \langle x, y, z(x, y) \rangle$$

$$\vec{r}_x = \langle 1, 0, \partial z / \partial x \rangle \quad \vec{r}_y = \langle 0, 1, \partial z / \partial y \rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \partial z / \partial x \\ 0 & 1 & \partial z / \partial y \end{vmatrix} = -\frac{\partial z}{\partial x} \hat{i} - \frac{\partial z}{\partial y} \hat{j} + \hat{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{(\partial z / \partial x)^2 + (\partial z / \partial y)^2 + 1}$$

$$\iint f[x, y, z(x, y)] \sqrt{(\partial z / \partial x)^2 + (\partial z / \partial y)^2 + 1} \, dx \, dy$$

$$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(\sin v)^2 + (-\cos v)^2 + u^2} = \sqrt{1 + u^2}$$

$$dS = |\vec{r}_u \times \vec{r}_v| \, du \, dv = \sqrt{1 + u^2} \, du \, dv$$

$$\iint_S f \, dS = \iint f \sqrt{1 + u^2} \, du \, dv$$

## Example

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