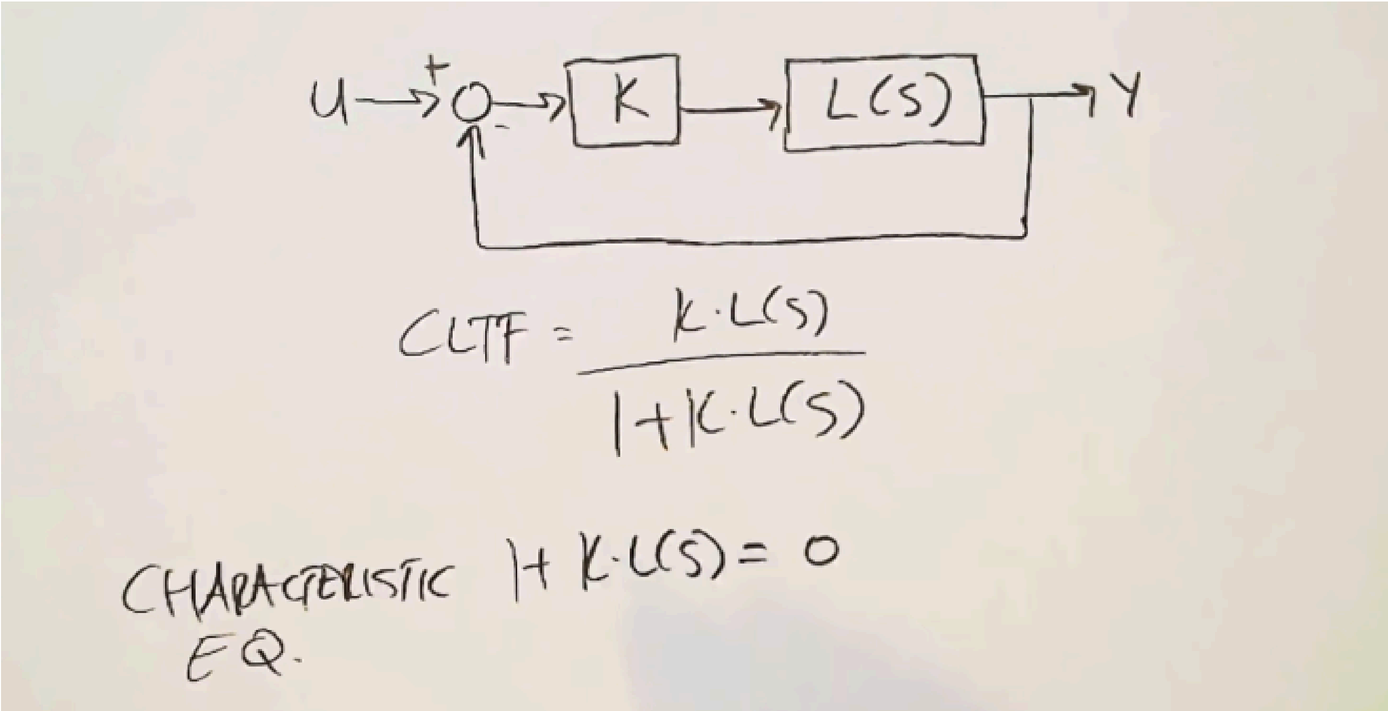
Root Locus Plots

Open Loop Vs Closed loop



$1 + k \frac{Z(s)}{P(s)}= 0$

P(s) + k Z(s) = 0

when k = 0,

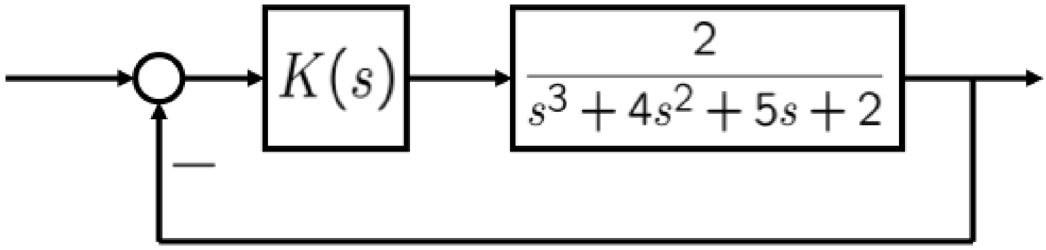
P(s) = 0.

when $k = \infty$

$P(s) +kZ(s) \approx kZ(s) = 0$

Z(s) = 0

Example 1



Design K(s) that stabilizes the closed-loop system for the following cases.

K(s) = K (constant)

\(1+ k\frac{2}{s^3+4s^2+5s+2} = 0 \)

$$\Rightarrow \quad s^3 + 4s^2 + 5s + 2 + 2K = 0$$

- **Routh array**

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s^3	1	5
s^2	4	$2+2K$
s^1	$\frac{18-2K}{4}$	
s^0	$2+2K$	

$$\Rightarrow \quad -1 < K < 9$$

Root Locus Plot Rules

1+kG(S) =0

Rule - the root locus is always symmetric with respect to the real axis.

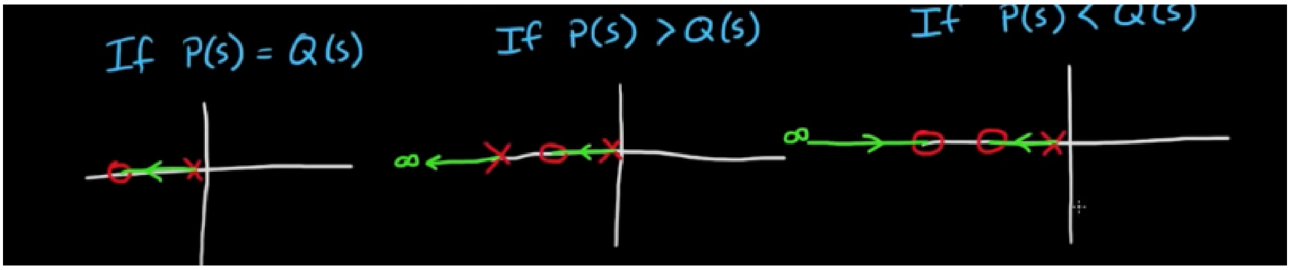
Rule -Total loci - max (p,2)

Rule -there are n lines (loci) where n is the degree of Q or P, which ever greater

Rule - total number of asymptotes = p - z

Rule - As k increase from 0 to \(\infty\) the roots move from the poles of G(s) to the zeros of G(s)

Comparing number of poles with number of zeros



Rule - when roots are complex they occur in conjugate pairs

Rule -at no time will the same root cross over its path

Rule - the portion of the real axis to the left of an odd number of open loop poles and zeros are part of the loci.

Rule

Lines leave (breakout) and enter (break in) the real axis at 90 degrees.

Rule

If there are not enough poles or zeros to make a pair then the extra lines go to or come from infinity.

Rule

Line goes to infinity along asymptotes

Angle of the asymptotes $\theta_{A} =\frac{(2q+1)180}{p-z}$

Centroid of the asymptotes = $\sum \frac{\text{finite poles} - \text{finite zeroes}}{p-z}$

p-z is the number of poles - number of zeroes --> number of lines that goes to infinity.

Rule

If there are at least two lines to infinity, then the sum of all the roots are constant.

Example 1

$\frac{s^2+s+1}{s^3+4s^2+ks+1}$

$s^3+4s^2+ks+1=0$. not in correct form

$1+k\frac{s}{s^3+4s^2+1}=0$.

$1+k\frac{P(s)}{Q(s)}=0$.

There are 3 lines (loci)

More poles that zeros

Example 2

Example 3

a) Open poles and zeros

Zeros: s = -6

Poles: s=-2, s=-4 , s=-6

b) Infinite poles and zeros

Number of zeros: 1

Number of poles: 3

We need two infinite zeros

c) Centroid of the asymptotes = $\sum \frac{\text{finite poles} - \text{finite zeroes}}{n-m}$

Centroid of the asymptotes = $\sum \frac{-2+-4+-9-(-6) }{3-1}=-4.5$

Angle of the asymptotes $\theta_{A} =\frac{(2q+1)180}{n-m}= \frac{(2q+1)180}{3-1}$

Example 4

a) Open poles and zeros

Poles: s=-1, s=-2 , s=-3

b) Infinite poles and zeros

Number of zeros: 0

Number of poles: 3

We need three infinite zeros

c) Centroid of the asymptotes = $\sum \frac{\text{finite poles} - \text{finite zeroes}}{n-m}$

Centroid of the asymptotes = $\sum \frac{-1-2-3 }{3}=-2$

Angle of the asymptotes $\theta_{A} =\frac{(2q+1)180}{n-m}= \frac{(2q+1)180}{3}$

$\frac{(2(0)+1)180}{3}= 60$

$\frac{(2(1)+1)180}{3}= 180$

$\frac{(2(2)+1)180}{3}= 300$

d) Imaginary axis crossing

$G_{\text{closed}} = \frac{k }{(s+1)(s+2)(s+3)+k}$

$G_{\text{closed}} = \frac{k}{s^3+6s^2+11s+6+k}$

Example 5

Centroid of the asymptotes = $\sum \frac{\text{finite poles} - \text{finite zeroes}}{n-m}$

Centroid of the asymptotes = $\sum \frac{-3-1 }{2}=-2$

Example 6

Centroid of the asymptotes = $\sum \frac{-3-1 -4 -5}{3-1}=-2.5$

Example 7

Assess whether the system is stable

Poles at -23.58 and -7.42

Example 8

$1 + G(s)H(s) = 0$

$1 + \frac{k}{S(S+2)} = 0$

$s^2+2s+k = 0$

a) k = 0

Poles at s = -2. and 0

b) k =1

Poles at s = -1

c) at k = 2

s = -1 + j, -1 - j

Example 9

Find the angle of departure

a). $G(S)H(S) = \frac{k}{s(s^2+2s+2)}$

b) $G(S)H(S) = \frac{k}{s(s+2)(s^2+2s+2)}$

c) $G(S)H(S) = \frac{k}{(s-1)(s^2+4s+2)}$

Example 10

\(\frac{1]{s(s+2)} \)

a) Poles: s=0, s = -2

b) Number of asymptotes = p - z = 2

c) Angle of asymptotes. = \(\frac{ (2x + 1)180}{p-z}\)

90, 270

d) Centroid of asymptotes = \(\frac{0+-2}{2} = -1 \)

e) Break away points

\(0 = 1 + k(s)h(s) \)

\(0 = 1 + \frac{k}{s(s+2)} \)

\(k = -s^2 -2s \)

\(\frac{dk}{ds} = -2s - 2\)

\(s = -1\)

Example 11

\(\frac{k(s+\frac{4}{3}}{s^2(s+12)}. \)

a) zeros: \(s = \frac{-4}{3}\)   Poles: s=0, s =0, s = -12

b) Number of Loci : max(3,1) = 3

b) Number of asymptotes = p - z = 3-1 = 2

c) Angle of asymptotes. = \(\frac{ (2x + 1)180}{p-z}\)

90, 270

d) Centroid of asymptotes = \(\frac{0+0-12 - (-1.33333)}{3-1} = -5.33 \)

e) Break away points

\(0 = 1 + k(s)h(s) \)

\(0 = 1 + \frac{k(s+\frac{4}{3}]{s^2(s+12)}. \)

\(k = -\frac{s^3+12s^2}{s+1.33} \)

\(\frac{dk}{ds} = 0 =\)

\(2s^3+16s^2+32s = 0\)

s = 0, s = -4

f) Angle of departure- not needed as all roots are on the real-axis.

Example 12

\(\frac{k}{s(s+1)(s+2)(s+3)}. \)

a) Poles: s=0, s =-1, s=-2 s = -3

b) Number of Loci : max(4,0) = 4

b) Number of asymptotes = p - z = 4-0 = 4

c) Angle of asymptotes. = \(\frac{ (2x + 1)180}{p-z}\)

45, 135,225, 315

d) Centroid of asymptotes = \(\frac{-1-1-2-3 - (0)}{4} = -1.5 \)

e)Angle of departure - no need as roots are on the real axis.

f) Break away points

\(0 = 1 + k(s)h(s) \)

\(0 = 1 + \frac{k}{s(s+1)(s+2)(s+3)}. \)

\(s^4 + 6k^3+11s^2+6s+k = 0\)

$k = -s^4 - 6k^3 -11s^2-6s$

$\frac{dk}{s} = -4s^3-18k^2-22s-6 = 0$

s = -0.4, -2.6

g) intersection with imaginary axis

Example 13

$\frac{k(s+2)}{(s+1)(s+3+2j)(s+3-2j)}.$

a) Poles: s=-2, s =-1, s=-3-2j s = -3+2j

b) Number of Loci : max(1,3) = 3

b) Number of asymptotes = p - z = 3-1 = 2

c) Angle of asymptotes. = $\frac{ (2x + 1)180}{p-z}$

90, 270

d) Centroid of asymptotes = $\frac{-3-3-1 + 2}{2} = -2.5$

e) Angle of departure $= 180 - [\Sigma P_{\theta} - \Sigma Z_{\theta}]$

Angle of departure $= 180 - [135+90-116.5] = 71.5$

Example 14

$s^2+2s+2+k(s+2) = 0$

Need inform $1+ G(s)H(s) = 0$

$1+k\frac{(s+2)}{s^2+2s+2} = 0$

$G(s)H(s) = k\frac{(s+2)}{s^2+2s+2}$

Zeroes, s = -2

Poles, s = -1-j. , s = -1+j

b) Number of Loci : max(1,2) = 2

b) Number of asymptotes = p - z = 2-1 = 1

c) Angle of asymptotes. = $\frac{ (2x + 1)180}{p-z}$

180

d) Centroid of asymptotes = \(\frac{-1-1+ 2}{2} = 0 \)

e) Angle of departure = 180 - [\Sigma P_{\theta} - \Sigma Z_{\theta}] \)

Angle of departure = 180 - [90-45] = 135.5 \)

f) Crosses the imaginary axis

\(s^2+2s+2+k(s+2) = 0 \)

\(k = -\frac{s+2}{s^2+2s+2}\)

\(\frac{dk}{ds} = 0 \)

\(s^2 + 4s+2\)

s = -0.58, -3.41

Example 15

\(G(S)(H(S) = \frac{k}{s(s+5)(s+10)} = 0 \)

Poles, s = 0 , s= -5, s= -10

b) Number of Loci :   max(0,3) = 3

b) Number of asymptotes = p - z = 3-0 = 3

c) Angle of asymptotes. = \(\frac{ (2x + 1)180}{p-z}\)

60,180,270

d) Centroid of asymptotes = \(\frac{0-5-10}{3} = 5 \)

e) Angle of departure = 180 - [\Sigma P_{\theta} - \Sigma Z_{\theta}] \)

Angle of departure = 180 - [90-45] = 135.5 \)

f) Break away point

1 + G(s)H(S) = 0

\(1+ \frac{k}{s(s+5)(s+10)} = 0\)

\(k = - (s^3+15s^2+50s)\)

\(\frac{dk}{ds} =   - (3s^2+30s^2+50)\)

\(s= -2.11, -7.88

Crosses the imaginary axis

$1+ \frac{k}{s(s+5)(s+10)} = 0$

$\frac{750-k}{15} \gt 0$

$75 \gt k \gt 0$

Auxillary Equation

$15s^2 +750 = 0$

$s= \pm j7.07$

g) Angle of departure

Not necessary as poles/zeros are on the real axis.

Example 16

$G(S)(H(S) = \frac{k}{s(s+2+2j)(s+2-2j)} = 0$

b) Number of Loci : max(0,3) = 3

b) Number of asymptotes = p - z = 3-0 = 3

c) Angle of asymptotes. = $\frac{ (2x + 1)180}{p-z}$

60,180,270

d) Centroid of asymptotes = $\frac{-2-2}{3} = \frac{-4}{3}$

e) Angle of departure = $180 - [\Sigma P_{\theta} - \Sigma Z_{\theta}]$

Angle of departure = $180 - [135+90] = -45$

f) Break away point

1 + G(s)H(S) = 0

$1+ \frac{k}{s(s+2+2j)(s+2-2j)} = 0$

$k = - (s^3+4s^2-8s)$

$\frac{dk}{ds} = - (3s^2+8s^2+50)$

$s= -1.33 \pm. j0.94$

Crosses the imaginary axis

$1+ \frac{k}{s(s+2+2j)(s+2-2j)} = 0$

$s^3+4s^2+8s+k= 0$

For stability

$32 \gt k \gt 0$

Auxiliary equation

4s^2 + 32 = 0

$s = \pm j2.82$

Example 16

$G(S)(H(S) = \frac{k(s+2)}{(s+1+1j)(s+1-1j)} = 0$

a) Number of Loci : max(1,2) = 2

b) Number of asymptotes = p - z = 2-1 = 1

) Angle of asymptotes. = $\frac{ (2x + 1)180}{p-z}$

180

d) Centroid of asymptotes = $\frac{-1-1+2}{3} = 0$

e) Angle of departure = $180 - [\Sigma P_{\theta} - \Sigma Z_{\theta}]$

Angle of departure = $180 - [135+90] = -45$

f) Break away point

1 + G(s)H(S) = 0

\(1 + \frac{k(s+2)}{(s+1+1j)(s+1-1j)} = 0 \)

\(k = \frac{s^2+2s+4}{s+2}\)

\(\frac{dk}{ds} = 0\)

s = 0, s = -4

Crosses the imaginary axis

1+G(S)H(S) = 0

\(s^2 + (2+k)s + (4+2k) = 0\)

For stability

\(k \gt -2 \)

But k needs to be positive

\(k \gt 0 \)

There is no intersection with the imaginary axis.

Angle of departure = 180 - [90-60] = 150 \)

References

https://www.youtube.com/watch?v=K19YgVJVP54 (https://www.youtube.com/watch?v=K19YgVJVP54)

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