Line Integrals

Background

line integral is denoted

$\int_C f(x,y)ds$

$\int_C f(x,y)\sqrt{\frac{dx}{dt}^2+\frac{dy}{dt}^2}dt$

$\int_C f(x,y)||r'(t)||dt$

For a line integral with respect to arc length if we change the direction, the integral does not change.
$\int_C f(x,y)ds = \int_{-C} f(x,y)ds$

Example

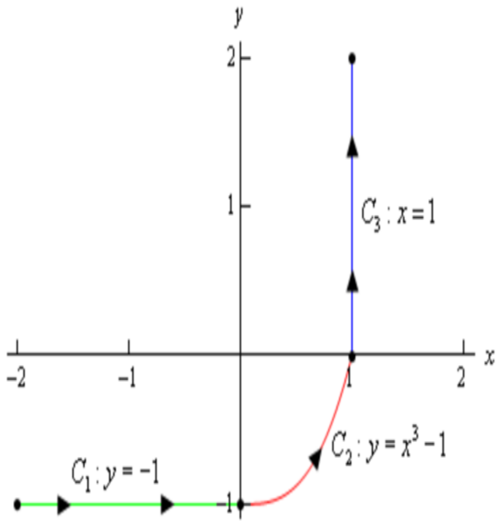
Evaluate

$\int_C xy^4 ds$ where C is the right half of the circle $x^2+y^2=r^2$

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} xy^4 ds = \frac{8192}{5}$

Example 2

Evaluate $\int_C 4x^3 ds$ where C is the curve shown below



$\int_{C_1} 4x^3 ds + \int_{C_2} 4x^3 ds + \int_{C_3} 4x^3 ds = -5.732$

Example 2

Evaluate $\int_C 4x^3 ds$ where C is the line segment from (-2,-1) to (2,1)

$\vec{r} = < -2 + 3t, -1 + 3t >$

$\int_0^1 4x^3 ds = -21.213$

Example 3

Evaluate $\int_C 4x^3 ds$ where C is the line segment from (2,1) to (-2,-1)

$\vec{r} = < 1 - 3t, 2 - 3t >$

$\int_0^1 4x^3 ds = -21.213$

Work and line integrals

Work = Force x distance = $\vec{F}. \Delta\vec{r}$

Along a trajectory C, work add up to .

$W = \int_C \vec{F}. d\vec{r}$

$W = \lim_{\Delta \to 0} \Sigma_i \vec{F}. \Delta\vec{r}_i$

$W = \lim_{\Delta \to 0} \Sigma_i \vec{F}. \frac{\Delta \vec{r}_i}{\Delta t} \Delta t$

$W = \int_{t_0}^{t_1} \vec{F}. \frac{d\vec{r}_i}{dt} dt$

Example

F = -yi + xj

C. $x = t$, $y = t^2$ $0 \le t \le 1$

Method 1

$W = \int_0^1 (-t^2 i + tj). (1i + 2tj) dt$

$W = \int_0^1 [-t^2 + 2t^2] dt = \frac{1}{3}$ <

Method 2

$F = Mi + Nj$, $d\vec{r} = dxi + dyj$

$W = \int_C \vec{F}.d\vec{r}$

$W = \int_C Mdx + Ndy$

dx = dt, dy = (2t)dt

$W = \int_C Mdt + N(2t)dt$

F = -yi + xj

$W = \int_C -t^2dt + 2t^2dt$

Geometric understanding

$d\vec{r} =< dx, dy >= \hat{T}ds$

$\frac{d\vec{r}}{dt} =< \frac{dx}{dt}, \frac{dy}{dt} >= \hat{T}\frac{ds}{dt}$

$W = \int_C \vec{F}.d\vec{r}$

$W = \int_C Mdx + Ndy = \int_C \vec{F}.\hat{T}ds$

Example

Circle of radius a at origin, counter clockwise.

$\vec{F} = xi + yj$

$\vec{F}$ is perpendicular to $\vec{T}$. Therefore

$\int_C \vec{F}.\hat{T}ds = 0$

Example

Circle of radius a at origin, counter clockwise.

$\vec{F} = -yi + xj$

$\vec{F}$ is parallel to $\vec{T}$. Therefore

$\int_C \vec{F}.\hat{T}ds = \int_C |\vec{F}|ds = \int_0^{2\pi} a^2ds = 2\pi a^2$