Implicit Function Theorem

Background

Function

Function is one-to-one or many to one. One-to-one function is one x values corresponds to one y values.

Projection of $\vec{a}$ onto $\vec{b}$

Projection = $|a|cos(\theta) = \frac{a.b}{|b|}$

Resolved vector of $\vec{a}$ onto $\vec{b}$

$\text{Resolved} = \text{Projection} \times \hat{b}$

$|\text{resolved}| = |\text{Projection}|$

Tangential planes

Plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$z_0 = f(x_0, y_0)$.

F(x,y,z)= z-f(x,y)

$\Delta F =< -f_x, -f_y, 1 >$

$\Delta F_{x0,y0,z0} =< -f_x(x_0, y_0), -f_y(x_0, y_0), 1 >$

Plane: $-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$

Chain Rule

w = w(x,y,z), x = x(t), y= y(t), z= z(t)

$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dx}{dt}$

$\frac{dw}{dt} = \nabla w. \frac{d\vec{F}}{dt}$

level curves

z = 2x + y

All these level curves will be lines e.g.

0 = 2x + y

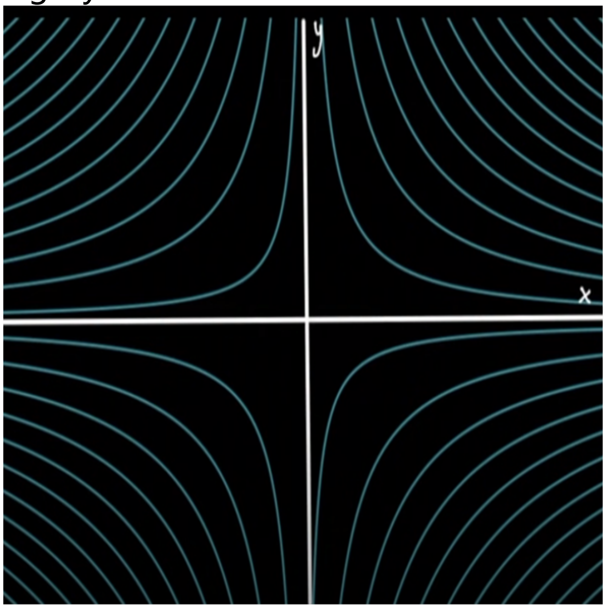
1 = 2x + y

$z = x^2 + y^2$ e.g $1 = x^2 + y^2$

Example

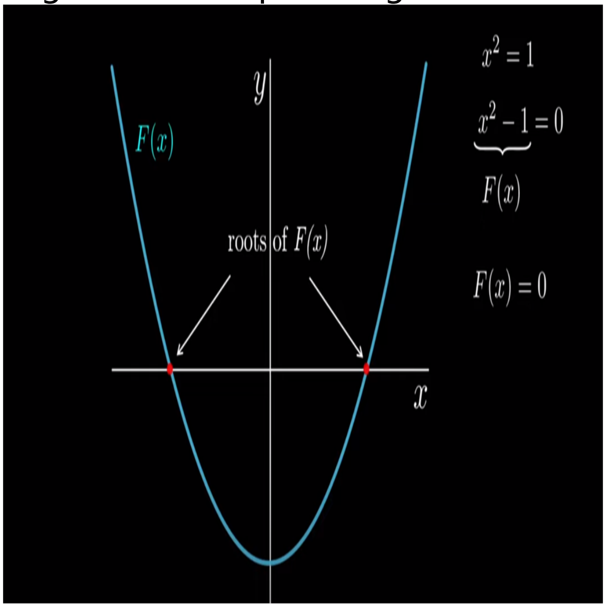
z= f(x,y) = xy

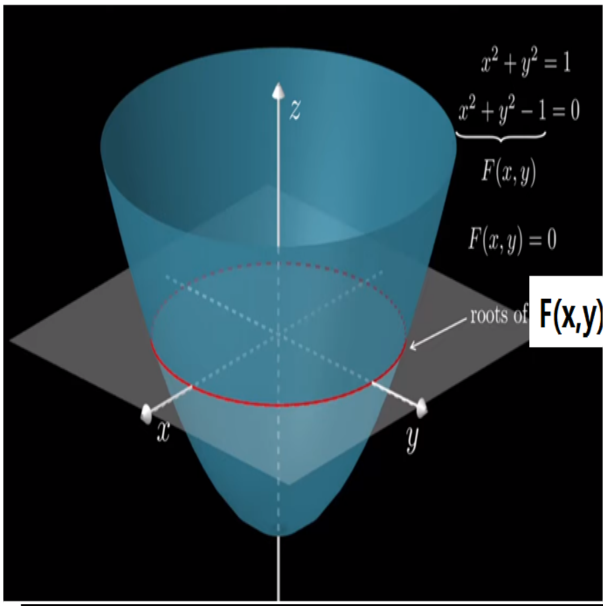
e.g. xy = 2

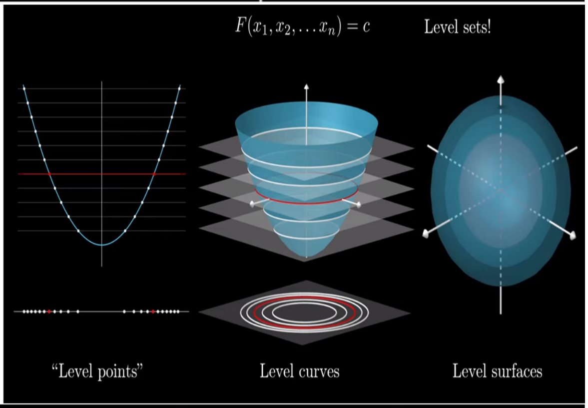


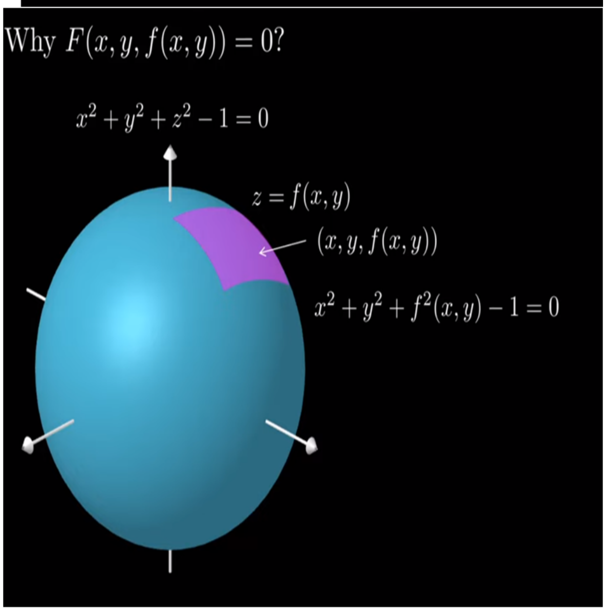
Level Surface

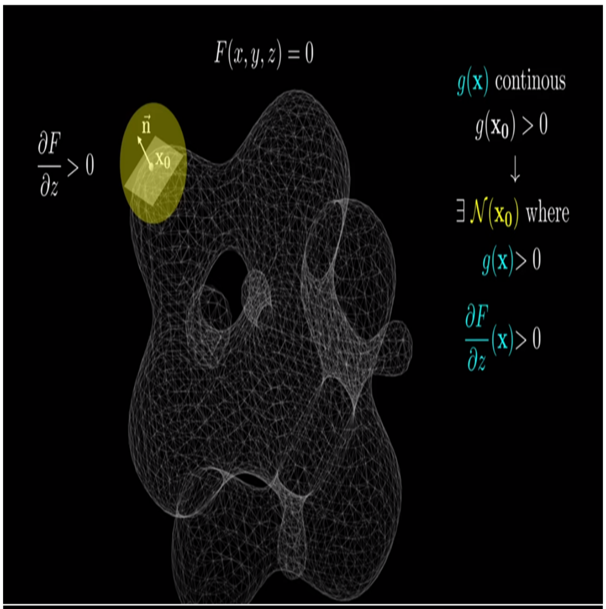
A surface S in the R^3 is called a level surface of f(x,y,z) if the value of f on every point S is some fixed constant. For example every body in a class room is the level surface of 37 degrees celsius- providing students do not have fever.

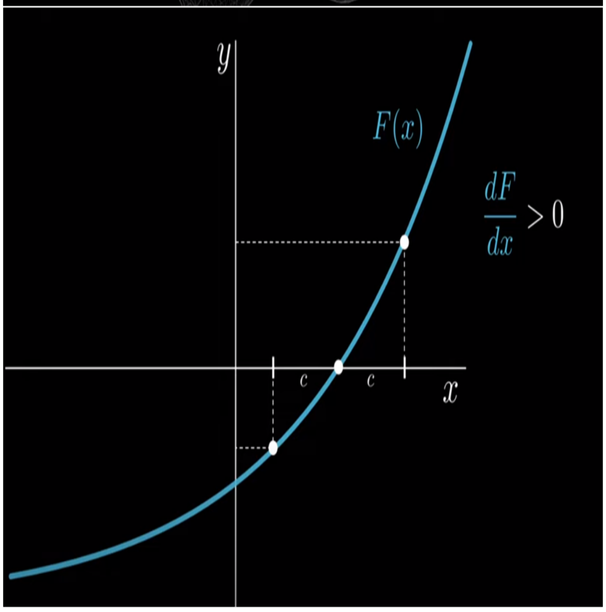



$x^2 + y^2 = 1$
$x^2 + y^2 - 1 = 0$
$F(x,y)$
$F(x,y) = 0$
roots of F(x,y)
z
x
y


$F(x_1, x_2, \dots x_n) = c$
Level sets!
"Level points"
Level curves
Level surfaces


Why $F(x, y, f(x, y)) = 0$?
$x^2 + y^2 + z^2 - 1 = 0$
$z = f(x, y)$
$(x, y, f(x, y))$
$x^2 + y^2 + f^2(x, y) - 1 = 0$





Gradient

Gradient is perpendicular to the level curves.

It points towards higher values.

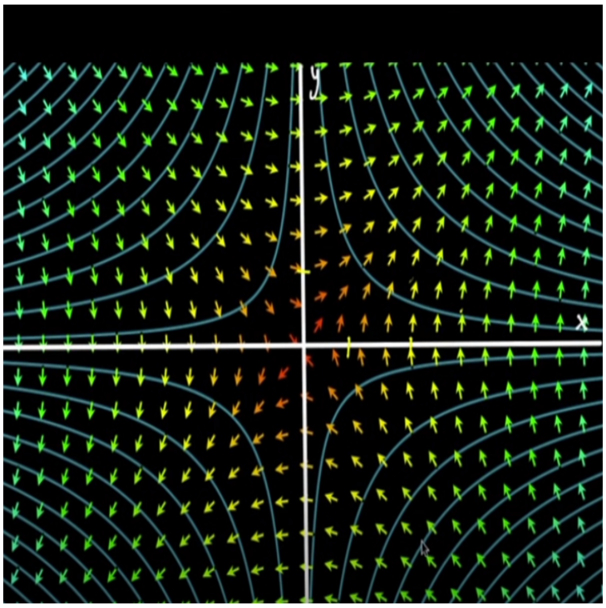
$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$$

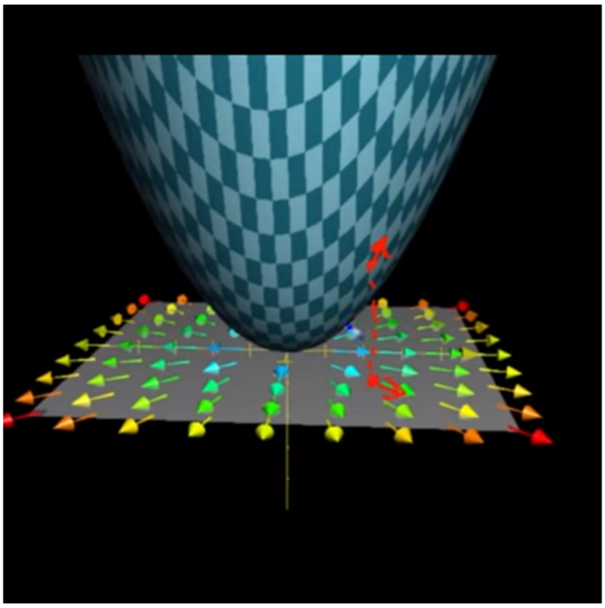
Example 1

$$f(x,y) = x^2 + y^2$$

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

Example 2

$z = f(x, y) = xy$ 



Example 2

$w = a_1x + a_2y + a_3z$,

$\nabla w =< a_1, a_2, a_3 >$

Level surface $c = a_1x + a_2y + a_3z$

This is a plane

The normal to the plane is the vector $< a_1, a_2, a_3 >$

This is the same as the gradient.

Example 3

$f(x, y, z) = x^2 + y^2 - z$

$0 = x^2 + y^2 - z$ - circular parabola.

$1 = x^2 + y^2 - z$ - circular parabola where the vertex is at -1.

Gradient is the direction of steepest ascent

Example 4

$w = x^2 + y^2$,

$\nabla w =< 2x, 2y, >$

Level curve $c = x^2 + y^2$

This is a circle where $\frac{dy}{dx} = \frac{-x}{y}$

The normal to the tangent is the vector $< x, y, >$

This is the same direction as the gradient.

Proof that normal to the tangent plane is the gradient

$\vec{r} = \vec{r}(t)$ stays on the level surface $w = F(x, y, z) = c$.

$\vec{r} =< x(t), y(t), z(t) >$

velocity vector is going to be tangential to the curve and also tangential to level surface (curve is inside the surface).

$\vec{v} = \frac{d\vec{r}}{dt}$ is tangential to the level surface $w = c$.

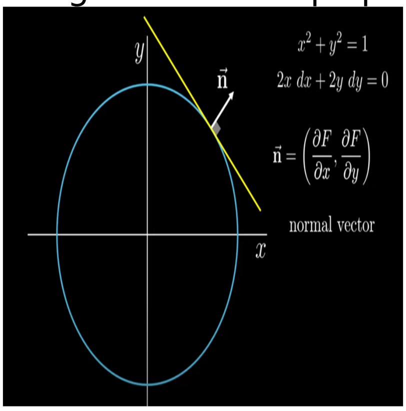
By the chain rule $\frac{dw}{dt} = \nabla w. \frac{d\vec{r}}{dt}$

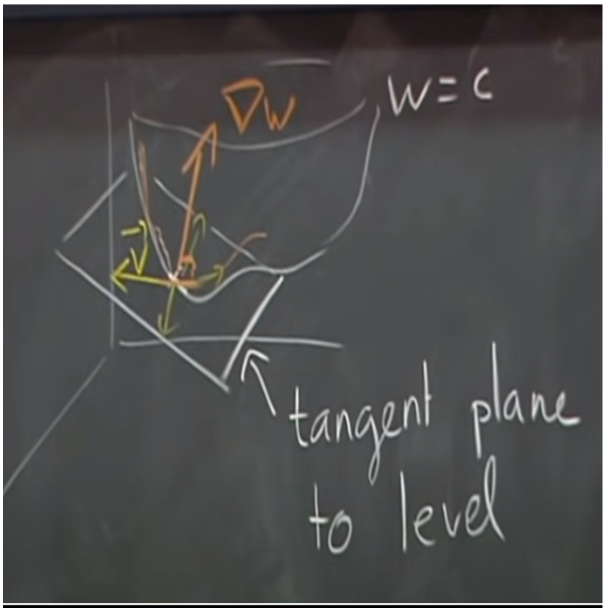
$\frac{dw}{dt} = \nabla w. \vec{v}$

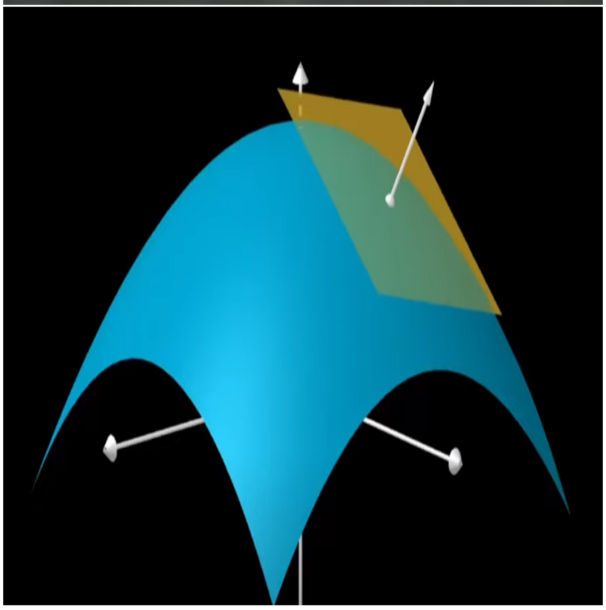
since w = c, therefore $\frac{dw}{dt} = 0$

Hence the velocity as the gradient are perpendicular to eachother

The gradient is also perpendicular to any vector on the tangential plane.







Finding Tangential plane to a surface

Level surface $x^2 + y^2 - z^2 = 4$ at (2,1,1)

$\nabla w = < 2x, 2y, -2z >$

Normal to tangential plane $< 4, 2, -2 >$

Tangential plane: $4x + 2y - 2z = 8$

Alternative method

$dw = 2xdx + 2ydy - 2zdz$

at (2,1,1) $dw = 4dx + 2dy - 2dz$

at (2,1,1) $\Delta W \approx 4\Delta x + 2\Delta y - 2\Delta z$

We stay on the level surface $\Delta W = 0$

$4(x-2) + 2(y-1) - 2(z-1) = 0$

Directional derivative

Fix a direction $\vec{u} =< u_1, u_2 >$ where $|\vec{u}| = 1$

$x(s) = x_0 + su_1$

$y(s) = y_0 + su_2$

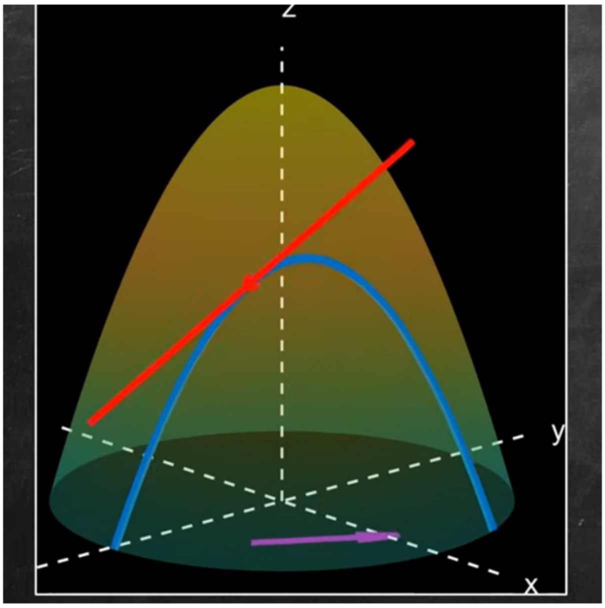
$D_{\vec{u}} f(x_0, y_0) = \frac{\lim_{s \to 0} f(x_0+su_1, y_0+su_2) - f(x_0,y_0)}{s}$

$D_{\vec{u}} f(x_0, y_0) = \frac{d}{ds}[f(x(s), y(s))]|_{s=0}$

$D_{\vec{u}} f(x_0, y_0) = f_x|_{(x_0,y_0)} \frac{dx}{ds} + f_y|_{(x_0,y_0)} \frac{dy}{ds}$

$D_{\vec{u}} f(x_0, y_0) = f_x|_{(x_0,y_0)} u_1 + f_y|_{(x_0,y_0)} u_2$

$D_{\vec{u}} f(x_0, y_0) = \nabla f|_{x_0,y_0} . \vec{u}$



Implicit Function Theorem

$F(x, y) \epsilon C^1$ in a neighbourhood of (x_0, y_0)

$F(x_0, y_0) = 0$

$\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$

df = $\frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial x} dx$

If these conditions are met then there is an explicit function y=f(x)

Implicit Function Theorem Examples

Example 1

$x^2 + y^2 = 1$

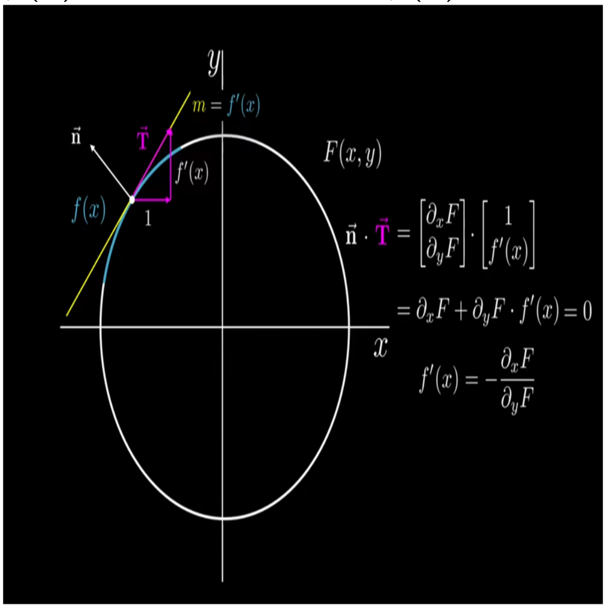
$F(x, y) = x^2 + y^2 - 1 = 0$

dF = $(2y)dy + (2x)dx$

$\frac{dy}{dx} = \frac{-x}{y}$

Interval is (-1,1)

$f(x) = \sqrt{1 - x^2}$ on I or $f(x) = -\sqrt{1 - x^2}$ on I



Example 2

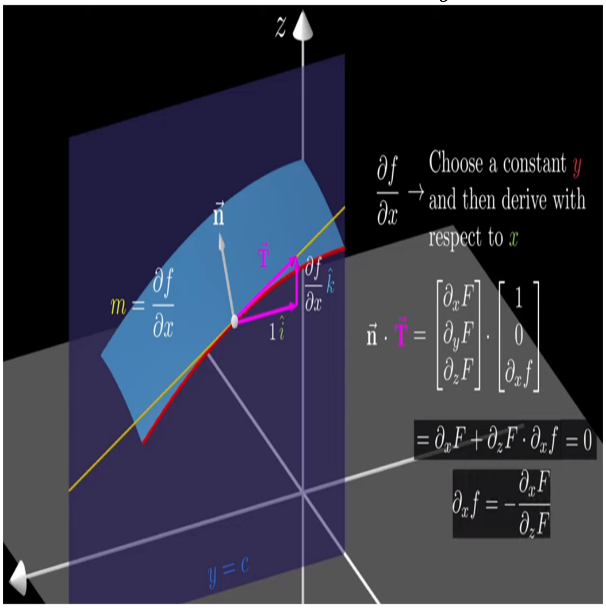
$\vec{n} =< \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} >$

z = f(x,y)

$m = \frac{dz}{dx} = \frac{\partial f}{\partial x}$

Tangent line $=< 1, \frac{\partial f}{\partial x}, 0 >$

$\vec{n}.\text{Tangent line} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial f}{\partial x} = 0$



Example 3

$F(x, y) = y^5 + y^3 + y + x = 0$

$F_y(x, y) = 5y^4 + 3y^2 + 1 > 0$

This function is strictly increasing

exactly one root

dF = $(5y^4 + 3y^2 + 1)dy + (x)dx$

$\frac{dy}{dx} = \frac{-1}{(5y^4+3y^2+1)}$

Generalisation- n + 1 coordinates

$\vec{x} = (x_1, x_2, \cdots, x_n)$

$F(\vec{x}, y) \epsilon C^1$ in a neighbourhood of $N_0(\vec{x_0}, y_0)$

$F(N_0) = 0$

$\frac{\partial f}{\partial y}(N_0) \neq 0$

If these conditions are met then there is an explicit function $y = f(\vec{x})$

$\frac{dy}{dx_i} = \frac{\partial f}{\partial x_i} = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial y}} \ (i = 1, 2, 3, \cdots, k)$

Example 4

$F(x, y, z) = 3x^2y - yz^2 - 4xz - 7 = 0$

We can show that near (-1,1,2) we can write y = f(x,z)

$F(-1,1,2)=0$

$\frac{\partial F}{\partial x}=6xy-4z$

$\frac{\partial F}{\partial y}=3x^2-z^2$

$\frac{\partial F}{\partial z}=2zy$

$\frac{dy}{dx}|_{(-1,2)}=\frac{\partial f}{\partial x}|_{(-1,2)}=-\frac{6xy-4z}{3x^2-z^2}|_{(-1,1,2)}=-14$

We can find y explicitly without the theorem.

$y=f(x,z)=\frac{4xz+7}{3x^2-z^2}$

Using the quotient rule

$\frac{\partial f}{\partial x}|_{(-1,2)}=-14$

Example 5

$F(x,y,z)=3x^2y-yz^2-4xz-7=0$

In this example, we can write z = f(x,y) explicitly by the quadratic formula

$z=\frac{4x\pm\sqrt{(-4x)^2-4(-y)(-7+3x^2y)}}{6x^2y}$

The theorem fails at $N_0(-1,1,2)$

$\frac{\partial F}{\partial z}=-2zy-4x$

$\frac{\partial F}{\partial z}(N_0)=0$

Example 6

$F(x,y)=(x-y)^3$

$F(x,y)=0$ therefore $y=x$

There is an explicit function at any point.

However at (0,0)

$\frac{\partial f}{\partial x}(0,0)=0$

$\frac{\partial f}{\partial y}(0,0)=0$

The theorem does not apply

Proof for two variables

$\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$

Case 1

$\frac{\partial F}{\partial y} > 0$

At a neighbourhood of (x_0, y_0)

$F(x_0, y)$ is strictly increasing in terms of y.

$F(x_0, y_0) = 0$

There exists a y_1 such that $F(x_0, y_1) > 0$

There exists a y_2 such that $F(x_0, y_2) < 0$

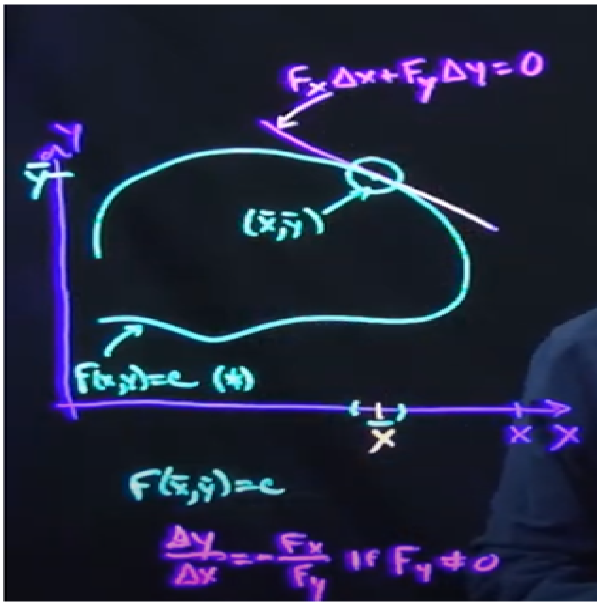
For every x near x_0

$F(x, y_1) > 0$

$F(x, y_2) < 0$

For such an x near x_0, since

$\frac{\partial f}{\partial y} > 0$, F(x,y) is increasing (as an increasing function of y)



Summary if $\frac{\partial F}{\partial y}(x_0, y_0) > 0$ therefore assuming (x,y) are near (x_0, y_0)

$\frac{\partial F}{\partial y}(x, y) > 0$

Therefore there exists a unique y such that $F(x,y) = 0$

f(x) is an implicit function with x as the domain.

This proves that y=f(x) exists and is unique proof of the formula for f'(x)

F(x,f(x)) = 0

By the chain rule

$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} f'(x) = 0$

$f'(x) = \frac{\frac{-\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$

The gradient is perpendicular to level surfaces

Suppose we have a function $g(x,y,z)\epsilon C^1$ at $M_0(x_0,y_0,z_0)$

$g(M_0) = g(x_0,y_0,z_0) = c_0$

Denote by S the level surface $g(x,y,z) = c_0$

Assume that $\nabla g(x_0,y_0,z_0) \neq 0$

Say for example that $\frac{\partial g}{\partial z} \neq 0$

$F(x,y,z) = g(x,y,z) - c_0$

$F(M_0) = 0$, $F\epsilon C^1$

$\frac{\partial F}{\partial z}(M_0) \neq 0$

By the implicit function theorem

$f\epsilon C^1$, $z_0 = f(x_0,y_0)$

$F(x,y,f(x,y)) = 0$ in a neighbourhood

$g(x,y,f(x,y)) = c_0$ near M_0

Hence near M_0 the level surface S is the graph of f(x,y).

The tangent plane at M_0

$z = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$

$z = f(x_0,y_0) + \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}(x_0,y_0)(x-x_0) + \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}(x_0,y_0)(y-y_0)$

$$z = f(x_0, y_0) + \frac{-\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}}(x_0, y_0)(x - x_0) + \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}(x_0, y_0)(y - y_0)$$

$$\frac{\partial g}{\partial x}(x - x_0) + \frac{\partial g}{\partial y}(y - y_0) + \frac{\partial g}{\partial z}(z - z_0) = 0$$

Hence the gradient of g is perpendicular to the tangential plane to S at M_0.

The gradient of g is the normal of the tangential plane.

Example

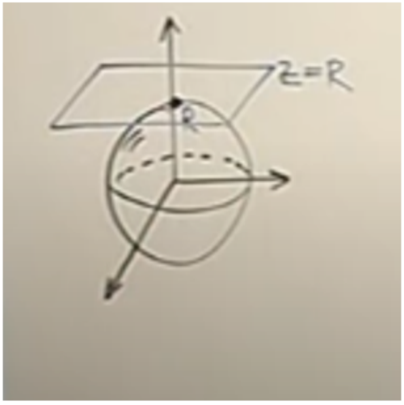
Find the tangential plane to the surface $x^2 + y^2 + z^2 = R^2$ at (0,0,R)

$g(x, y, z) = x^2 + y^2 + z^2 - R^2$

$\nabla g =< 2x, 2y, 2z >$

Normal of the tangent $< 0, 0, 2R >$

2R z = d

Tangent plane: z = R 

Example 6

$x^2 + y^2 + z^2 = sin(zy)$

$F(x, y, z) = x^2 + y^2 + z^2 - sin(zy)$

$\frac{\partial z}{\partial x} = -\frac{F_x}{F_y}$

$\frac{\partial z}{\partial x} = -\frac{2x}{2z - ycos(zy)}$

$\frac{\partial z}{\partial y} = -\frac{F_x}{F_z}$

$\frac{\partial z}{\partial y} = -\frac{2y - zcos(zy)}{2z - ycos(zy)}$

Example 7

$x^2 + y^4 + z^3 + 3xy^2 = 8$

$F(x,y) = x^2 + y^4 + z^3 + 3xy^2 - 8$

$F_x(x,y) = 2x + 3y^2$

$F_y(x,y) = 4y^3 + 6xy$

$F_z(x,y) = -3z^2$

$\frac{\partial z}{\partial x} = -\frac{2x+3y^2}{-3z^2}$

$\frac{\partial z}{\partial y} = -\frac{4y^3+6xy}{-3z^2}$

Method 2- implicit differentiation

$2x + 3z^2\frac{\partial z}{\partial x} + 3y^2 = 8$

$\frac{\partial z}{\partial x} = -\frac{2x+3y^2}{-3z^2}$

Example 8

$xy^3 + x^2z^2 = 6$

$F(x,y) = xy^3 + x^2z^2 - 6$

$F_x(x,y) = y^3 + 2xz^2$

$F_y(x,y) = 3xy^2$

$F_z(x,y) = 2zx^2$

$\frac{\partial z}{\partial x} = -\frac{y^3+2xz^2}{2zx^2}$

$\frac{\partial z}{\partial y} = -\frac{3xy^2}{2zx^2}$